NUMERICAL OPTIMIZATION OF KERNEL BASED IMAGE DERIVATIVES

Dirk-Jan Kroon

University of Twente, Enschede
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ABSTRACT

This short-paper gives results obtained with existing derivative kernels such as the Sobel-operator and Scharr kernel, and introduces orientation optimized kernels. The optimized kernels are found using a minimizer on the absolute angular errors of an image containing circular patterns with varying spatial frequencies. Results show that a $3 \times 3$ truncated area sampled Gaussian is as rotation invariant as any other kernel, it is only a matter of choosing the right sigma. When using $5 \times 5$ sized derivative kernels, the Gaussian derivatives are outperformed in angle accuracy by angle error optimized kernels.

Index Terms— Derivatives, Rotation, Optimization

1. INTRODUCTION

In anisotropic diffusion filtering rotational invariance of the discretized diffusion filtering scheme is wanted. Most schemes contain first order and second order spatial derivatives. We try to obtain the best rotational image derivative kernels with a small local footprint.

2. THE TEST IMAGE

As a test image we uses an image with sizes $255 \times 255$ containing the following function:

$$I = \sin(x^2 + y^2) \quad (1)$$

With $x$ and $y$ running from $-10$ to $10$, see picture 1. The first order derivatives are:

$$I_x = 2xcos(x^2 + y^2)$$
$$I_y = 2ycos(x^2 + y^2) \quad (2)$$

The second order derivatives are:

$$I_{xx} = 2cos(x^2 + y^2) - 4x^2sin(x^2 + y^2)$$
$$I_{yy} = 2cos(x^2 + y^2) - 4y^2sin(x^2 + y^2)$$
$$I_{xy} = -4xysin(x^2 + y^2) \quad (3)$$

Fig. 2. This figure shows the angles calculated with first order derivatives (a), and the image angles calculated from the eigenvectors of the hessian constructed from second order derivatives (b).

3. ANGLE FROM DERIVATIVES

The angle $\alpha$ of first order image derivatives can be calculated, with the following equation:

$$\alpha = \arctan \left( \frac{I_x(x,y)}{I_y(x,y)} \right) \quad (4)$$

Second order derivatives are often used to construct the Hessian matrix to calculate the direction of an image structure such as a line or vessel. The Hessian for ever image pixel coordinate $(i,j)$,

$$H(i,j) = \begin{bmatrix} I_{xx}(i,j) & I_{xy}(i,j) \\ I_{xy}(i,j) & I_{yy}(i,j) \end{bmatrix} \quad (5)$$

Eigenvector analysis of the Hessian $H(i,j)$ can be used to find the eigenvectors $v_1 = [v_{1x}, v_{1y}]$ and $v_2 = [v_{2x}, v_{2y}]$ which give the main orientations the image structure beneath a pixel coordinate, and than the angle is calculated with the following equation.

$$\beta = \arctan \left( \frac{v_{2x}}{v_{2y}} \right) \quad (6)$$

For the angles obtained from the first order and second order derivatives of the test image see figure 2.

4. FIRST ORDER DERIVATIVES

First we will try the commonly used Sobel operator on our test image, this kernel consist of a 1D smoothing kernel, mult-
Fig. 1. Test image, and it’s analytical first order \( I_x, I_y \) and second order \( I_{xx}, I_{xy}, I_{yy} \) derivatives.

Fig. 3. Absolute error in angle from the first order derivatives approximated by the Sobel kernel(a), and by the Scharr kernel(b).

Fig. 4. Area used for minimalization of the absolute angle error, and the absolute angle error after optimization

Previous kernels already very small angle approximation errors in vertical, horizontal and diagonal directions. So we optimize on the absolute error in the area \( A \) of the other directions, see picture 4.

As an optimizer we use an quasi Newton optimizer from called FMINLBFGS. The found ratio between the kernel values \( p_1 \) and \( p_2 \) is 3.5887, which is very close to ratio of Scharr’s kernel \( 3^2 \). If we write the optimized kernel in rounded values we get:

\[
H_x = \begin{bmatrix}
+17 & +61 & +17 \\
0 & 0 & 0 \\
-17 & -61 & -17
\end{bmatrix}
\]

This matrix is equal to that of a truncated Gaussian derivative of \( \sigma = 0.548 \), (made with area sampling of the kernel to prevent large aliasing errors).

Instead of using a \( 3 \times 3 \) derivative kernel we can also use a \( 5 \times 5 \) kernel. The truncated Gaussian derivative with optimized sigma \( \sigma = 0.6769 \) to give the minimal angle error is:

\[
H_x = \begin{bmatrix}
0.0007 & 0.0108 & 0.0270 & 0.0108 & 0.0007 \\
0.0053 & 0.0863 & 0.2150 & 0.0863 & 0.0053 \\
0 & 0 & 0 & 0 & 0 \\
-0.0053 & -0.0863 & -0.2150 & -0.0863 & -0.0053 \\
-0.0007 & -0.0108 & -0.0270 & -0.0108 & -0.0007
\end{bmatrix}
\]

The kernel found with the Matlab optimizer FMIN-
from the angle error minimalization is almost equal to a $\beta$ luterror between analytical angle to find the

Instead of using those kernels, we can also use an optimizer

kernels are

cated Gaussian derivatives.

The resulting absolute angle errors with these kernels are shown in figure 5. The optimized kernel outperforms the truncated Gaussian derivatives.

5. SECOND ORDER DERIVATIVES

The $3 \times 3$ kernels needed to calculate the second order derivative approximations $J_{xx}, J_{xy}, J_{yy}$, can be derived from a truncated Gaussian function, for instance with $\sigma = 0.8$ the kernels are

$$H_{xx} = \begin{bmatrix} 0.0007 & 0.0052 & 0.0370 & 0.0052 & 0.0007 \\ 0.0037 & 0.1187 & 0.2589 & 0.1187 & 0.0037 \\ 0 & 0 & 0 & 0 & 0 \\ -0.0037 & -0.1187 & -0.2589 & -0.1187 & -0.0037 \\ -0.0007 & -0.0052 & -0.0370 & -0.0052 & -0.0007 \end{bmatrix}$$

(13)

The absolute angle error, with truncated gaussian kernel with optimized sigma, and the error with kernel values optimized for angle error with optimizer.

Fig. 5. The absolute angle error, with truncated gaussian kernel with optimized sigma, and the error with kernel values optimized for angle error with optimizer.

The results show that using a $3 \times 3$ matrix to calculate the second order derivatives. With optimization for the kernel values we find the kernel below. We compare its performance with the $5 \times 5$ truncated second order Gaussian derivative, see figure 7.

$$H_{xx} = \begin{bmatrix} 0.0033 & 0.0435 & 0.0990 & 0.0435 & 0.0033 \\ 0.0045 & 0.0557 & 0.1009 & 0.0557 & 0.0045 \\ -0.0156 & -0.2032 & -0.4707 & -0.2032 & -0.0156 \\ 0.0045 & 0.0557 & 0.1009 & 0.0557 & 0.0045 \\ 0.0033 & 0.0435 & 0.0990 & 0.0435 & 0.0033 \end{bmatrix}$$

(18)

$$H_{xy} = \begin{bmatrix} 0.0034 & 0.0211 & 0 & -0.0211 & -0.0034 \\ 0.0211 & 0.1514 & 0 & -0.1514 & -0.0211 \\ 0 & 0 & 0 & 0 & 0 \\ -0.0211 & -0.1514 & 0 & 0.1514 & 0.0211 \\ -0.0034 & -0.0211 & 0 & 0.0211 & 0.0034 \end{bmatrix}$$

(19)

Fig. 6. Angles calculated from the eigenvectors of approximated hessian, derivatives from truncated gaussian function with $\sigma = 0.8$ (a), derivatives from gaussian function with optimized sigma(b) $\sigma = 0.977$, derivatives kernel optimized for smallest angle errors(c)

Fig. 7. The absolute angle error, with truncated gaussian kernel with optimized sigma, and the error with kernel values optimized for angle error with optimizer.

Obviously we also can use an $5 \times 5$ matrix to calculate the second order derivatives. With optimization for the kernel values we find the kernel below. We compare its performance with the $5 \times 5$ truncated second order Gaussian derivative, see figure 7.

$$H_{xx} = \begin{bmatrix} 0.0033 & 0.0435 & 0.0990 & 0.0435 & 0.0033 \\ 0.0045 & 0.0557 & 0.1009 & 0.0557 & 0.0045 \\ -0.0156 & -0.2032 & -0.4707 & -0.2032 & -0.0156 \\ 0.0045 & 0.0557 & 0.1009 & 0.0557 & 0.0045 \\ 0.0033 & 0.0435 & 0.0990 & 0.0435 & 0.0033 \end{bmatrix}$$

(18)

$$H_{xy} = \begin{bmatrix} 0.0034 & 0.0211 & 0 & -0.0211 & -0.0034 \\ 0.0211 & 0.1514 & 0 & -0.1514 & -0.0211 \\ 0 & 0 & 0 & 0 & 0 \\ -0.0211 & -0.1514 & 0 & 0.1514 & 0.0211 \\ -0.0034 & -0.0211 & 0 & 0.0211 & 0.0034 \end{bmatrix}$$

(19)

6. DISCUSSION

The results show that using a $3 \times 3$ truncated area sampled Gaussian is as rotation invariant as any other kernel, it is only a matter of choosing the right sigma. When using $5 \times 5$ derivative kernels the Gaussian derivatives are outperformed in angle error by optimized kernels, made by finding the kernel values with an angle error minimizer.